The universal Askey-Wilson algebra

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This talk concerns an algebra $\Delta$ called the **Universal Askey-Wilson algebra**.

As we will see, $\Delta$ is related to:

- Leonard pairs and Leonard triples of QRacah type
- $Q$-polynomial distance-regular graphs of QRacah type
- The modular group $\text{PSL}_2(\mathbb{Z})$
- The equitable presentation of the quantum group $U_q(\mathfrak{sl}_2)$
- The double affine Hecke algebra of type $(C_1^\vee, C_1)$
We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be tridiagonal.

The following matrices are tridiagonal.

\[
\begin{pmatrix}
2 & 3 & 0 & 0 \\
1 & 4 & 2 & 0 \\
0 & 5 & 3 & 3 \\
0 & 0 & 3 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 3 & 0 & 0 \\
0 & 4 & 2 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 1 & 5
\end{pmatrix}.
\]

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is irreducible. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.
We now define a Leonard pair. From now on $\mathbb{F}$ will denote a field.

**Definition (Terwilliger 1999)**

Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a **Leonard pair** on $V$, we mean a pair of linear transformations $A : V \rightarrow V$ and $B : V \rightarrow V$ such that:

1. There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal and the matrix representing $B$ is irreducible tridiagonal.

2. There exists a basis for $V$ with respect to which the matrix representing $B$ is diagonal and the matrix representing $A$ is irreducible tridiagonal.
In summary, for a Leonard pair $A, B$

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<thead>
<tr>
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<tr>
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The term **Leonard pair** is motivated by a 1982 theorem of **Doug Leonard** concerning the QRacah polynomials and some related polynomials in the Askey scheme.

For a detailed version of Leonard’s theorem see the book

Here is an example of a Leonard pair.

Fix an integer $d \geq 3$.
Pick nonzero scalars $a, b, c, q$ in $\mathbb{F}$ such that

(i) $q^{2i} \neq 1$ for $1 \leq i \leq d$;
(ii) Neither of $a^2$, $b^2$ is among $q^{2d-2}$, $q^{2d-4}$, $\ldots$, $q^{2-2d}$;
(iii) None of $abc$, $a^{-1}bc$, $ab^{-1}c$, $abc^{-1}$ is among $q^{d-1}$, $q^{d-3}$, $\ldots$, $q^{1-d}$. 
Define

$$
\theta_i = a q^{2i-d} + a^{-1} q^{d-2i},
$$

$$
\theta_i^* = b q^{2i-d} + b^{-1} q^{d-2i}
$$

for $0 \leq i \leq d$ and

$$
\varphi_i = a^{-1} b^{-1} q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})
$$

$$
(q^{-i} - abc q^{i-d-1})(q^{-i} - abc^{-1} q^{i-d-1})
$$

for $1 \leq i \leq d$. 
Define

\[ A = \begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \cdots & 0 \\ 1 & 1 & \theta_2 & \cdots & \theta_d \\ \theta_0^* & \theta_1^* & \varphi_2 & \cdots & \cdots \\ \theta_1^* & \varphi_1 & \theta_2^* & \cdots & \varphi_d \\ 0 & 1 & \varphi_d & \theta_d^* & \varphi_d \\ \end{pmatrix} \]

\[ B = \begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \cdots & 0 \\ 1 & 1 & \theta_2 & \cdots & \theta_d \\ \theta_0^* & \theta_1^* & \varphi_2 & \cdots & \cdots \\ \theta_1^* & \varphi_1 & \theta_2^* & \cdots & \varphi_d \\ 0 & 1 & \varphi_d & \theta_d^* & \varphi_d \\ \end{pmatrix} \]
Then the pair $A, B$ is a Leonard pair on the vector space $V = \mathbb{F}^{d+1}$.

A Leonard pair of this form is said to have **QRacah type**.

This is the most general type of Leonard pair.
Hau-wen Huang (former student of Chih-wen Weng in the Department of Applied Math, National Chiao Tung University, Taiwan) has proven the following beautiful theorem about the Leonard pairs of QRacah type.
The $\mathbb{Z}_3$-symmetric Askey-Wilson relations

**Theorem (Hau-wen Huang 2011)**

Referring to the above Leonard pair $A, B$ of QRacah type, there exists an element $C$ such that

\[
\begin{align*}
A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}} \\
B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}} \\
C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}
\end{align*}
\]

The above equations are called the $\mathbb{Z}_3$-symmetric Askey-Wilson relations.
In the previous example the $\mathbb{Z}_3$-symmetry involving $A, B, C$ suggests that we should consider Leonard triples along with Leonard pairs.

The notion of a Leonard triple is due to Brian Curtin and defined as follows.

**Definition (Brian Curtin 2007)**

By a **Leonard triple** on $V$ we mean an ordered triple of linear transformations $(A, B, C)$ in $\text{End}(V)$ such that for each $\phi \in \{A, B, C\}$ there exists a basis for $V$ with respect to which the matrix representing $\phi$ is diagonal and the matrices representing the other two linear transformations are irreducible tridiagonal.
In summary, for a Leonard triple $A$, $B$, $C$

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<td>irreducible tridiagonal</td>
</tr>
<tr>
<td>basis 3</td>
<td>irreducible tridiagonal</td>
<td>irreducible tridiagonal</td>
<td>diagonal</td>
</tr>
</tbody>
</table>
For a moment let us return to our Leonard pair $A, B$ of QRacah type.

Consider the element $C$ from the $\mathbb{Z}_3$-symmetric Askey-Wilson relations.

Huang has found necessary and sufficient conditions on $C$ for the triple $A, B, C$ to be a Leonard triple.

This is explained in the next two theorems.
Theorem (Hau-wen Huang 2011)

The roots of the characteristic polynomial of $C$ are $\{\theta_i^\epsilon\}_{i=0}^d$, where

$$\theta_i^\epsilon = cq^{2i-d} + c^{-1}q^{d-2i} \quad (0 \leq i \leq d)$$
Leonard pairs and Leonard triples, cont.

Theorem (Hau-wen Huang 2011)

The following (i)–(iii) are equivalent.

(i) The triple $A, B, C$ is a Leonard triple;

(ii) $\{\theta_i^\xi\}_{i=0}^d$ are mutually distinct;

(iii) $c^2$ is not among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.

The above Leonard triple is said to have QRacah type.
In 1992 Alexei Zhedanov introduced the Askey-Wilson algebra $AW=AW(3)$ and used it to describe the Askey-Wilson polynomials.

Essentially, $AW$ is the algebra defined by three generators $A, B, C$ subject to the $\mathbb{Z}_3$-symmetric Askey-Wilson relations (The original definition was somewhat different).

The algebra $AW$ is defined using four parameters $a, b, c, q$. 
We now define a central extension of AW, called the **universal Askey-Wilson algebra** and denoted $\Delta$.

The algebra $\Delta$ involves just one parameter $q$.

The algebra $\Delta$ is defined as follows.

For the rest of the talk, $q$ denotes a nonzero scalar in $\mathbb{F}$ such that $q^4 \neq 1$. 

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The universal Askey-Wilson algebra
The universal Askey-Wilson algebra

Definition (Ter 2011)

Define an $\mathbb{F}$-algebra $\Delta = \Delta_q$ by generators and relations in the following way. The generators are $A, B, C$. The relations assert that each of

\[ A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \]

is central in $\Delta$. We call $\Delta$ the universal Askey-Wilson algebra.
By construction, each Askey-Wilson algebra $\mathcal{A}_W$ is a homomorphic image of $\Delta$.

By construction, each Leonard pair or triple of QRacah type can be viewed as a $\Delta$-module.
We now briefly relate $\Delta$ to $Q$-polynomial distance-regular graphs.

Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$ and distance matrices $\{A_i\}_{i=0}^D$.

Assume $\Gamma$ has a $Q$-polynomial ordering $\{E_i\}_{i=0}^D$ of its primitive idempotents.

Assume that the $Q$-polynomial structure has QRacah type; this means (in the notation of Bannai/Ito) type I with each of $s, s^*$ nonzero.
Fix a vertex $x$ of $\Gamma$. For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the dual distance matrix of $\Gamma$ that corresponds to $E_i$ and $x$.

Assume that each irreducible $T$-module is thin. Here $T = T(x)$ is the subconstituent algebra of $\Gamma$ with respect to $x$, generated by $A_1$ and $A_1^*$.

**Theorem (Arjana Zitnik, Ter, in preparation)**

With the above assumptions and notation, there exists a surjective algebra homomorphism $\Delta \rightarrow T$ that sends the generator $A$ to a linear combination of $I$, $A_1$ and the generator $B$ to a linear combination of $I$, $A_1^*$. 

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The universal Askey-Wilson algebra
Three central elements of $\Delta$

We now describe $\Delta$ from a ring theoretic point of view.

**Definition**

Define elements $\alpha, \beta, \gamma$ of $\Delta$ such that

\[
A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}},
\]
\[
B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}},
\]
\[
C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}.
\]

Note that each of $\alpha$, $\beta$, $\gamma$ is central in $\Delta$. 

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The universal Askey-Wilson algebra
A basis for $\Delta$

**Theorem (Ter 2011)**

The following is a basis for the $\mathbb{F}$-vector space $\Delta$:

$$A^i B^j C^k \alpha^r \beta^s \gamma^t$$

$i, j, k, r, s, t \in \mathbb{N}$.

We proved this using the Bergman Diamond Lemma.
An action of $\text{PSL}_2(\mathbb{Z})$ on $\Delta$

Recall that the modular group $\text{PSL}_2(\mathbb{Z})$ has a presentation by generators $p, s$ and relations $p^3 = 1, s^2 = 1$.

Our next goal is to show that $\text{PSL}_2(\mathbb{Z})$ acts on $\Delta$ as a group of automorphisms.

Strategy: identify two automorphisms of $\Delta$ that have orders 3 and 2.

By construction $\Delta$ has an automorphism that sends

$$A \mapsto B \mapsto C \mapsto A.$$ 

This automorphism has order 3.

To find an automorphism of $\Delta$ that has order 2, we use another presentation for $\Delta$. 

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Paul Terwilliger  The universal Askey-Wilson algebra
The algebra $\Delta$ has a presentation by generators $A, B, \gamma$ and relations

\[
A^3 B - [3]_q A^2 BA + [3]_q ABA^2 - BA^3 = -(q^2 - q^{-2})^2(AB - BA),
\]
\[
B^3 A - [3]_q B^2 AB + [3]_q BAB^2 - AB^3 = -(q^2 - q^{-2})^2(BA - AB),
\]
\[
A^2 B^2 - B^2 A^2 + (q^2 + q^{-2})(BABA - ABAB)
\]
\[
= -(q - q^{-1})^2(AB - BA)\gamma,
\]
\[
\gamma A = A\gamma, \quad \gamma B = B\gamma.
\]

Here $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

The first two relations above are the \textbf{tridiagonal relations}.
An automorphism of $\Delta$ that has order 2

By the alternate presentation $\Delta$ has an automorphism that swaps $A$, $B$ and fixes $\gamma$.

This automorphism has order 2.
The group $\text{PSL}_2(\mathbb{Z})$ acts on $\Delta$ as a group of automorphisms in the following way:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
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<tr>
<td>$p(u)$</td>
<td>$B$</td>
<td>$C$</td>
<td>$A$</td>
<td>$\beta$</td>
<td>$\gamma$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$s(u)$</td>
<td>$B$</td>
<td>$A$</td>
<td>$C + \frac{AB-BA}{q-q^{-1}}$</td>
<td>$\beta$</td>
<td>$\alpha$</td>
<td>$\gamma$</td>
</tr>
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This action is faithful.
The Casimir element $\Omega$ of $\Delta$

Shortly we will describe the center $Z(\Delta)$.

To do this we introduce a certain element $\Omega \in \Delta$ called the Casimir element.

**Definition**

Define

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma.$$ 

We call $\Omega$ the **Casimir element** of $\Delta$. 

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The universal Askey-Wilson algebra
The Casimir element $\Omega$ is central

**Theorem (Ter 2011)**

The Casimir element $\Omega$ is contained in $\mathbb{Z}(\Delta)$.

Moreover $\Omega$ is fixed by everything in $\text{PSL}_2(\mathbb{Z})$. 
We are going to show that $Z(\Delta)$ is generated by $\Omega, \alpha, \beta, \gamma$ provided that $q$ is not a root of unity.

To this end we display a basis for $\Delta$ that involves $\Omega$.

**Lemma (Ter 2011)**

The following is a basis for the $\mathbb{F}$-vector space $\Delta$:

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t$$

$i, j, k, \ell, r, s, t \in \mathbb{N}$, $ijk = 0$. 
Corollary (Ter 2011)

The elements $\Omega, \alpha, \beta, \gamma$ are algebraically independent over $\mathbb{F}$. 
We now describe the center $Z(\Delta)$.

**Theorem (Ter 2011)**

Assume that $q$ is not a root of unity. Then the algebra $Z(\Delta)$ is generated by $\Omega, \alpha, \beta, \gamma$.

Moreover $Z(\Delta)$ is isomorphic to a polynomial algebra in four variables.
Our next goal is to explain how $\Delta$ is related to the quantum group $U_q(\mathfrak{sl}_2)$.

**Definition**

The $\mathbb{F}$-algebra $U = U_q(\mathfrak{sl}_2)$ is defined by generators $e, f, k^{\pm 1}$ and relations

\[
k k^{-1} = k^{-1} k = 1,
ke = q^2 ek, \quad kf = q^{-2} fk,
ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.
\]

We call $e, f, k^{\pm 1}$ the **Chevalley generators** for $U$.  

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The universal Askey-Wilson algebra
Irreducible modules for $U_q(\mathfrak{sl}_2)$

We review the finite-dimensional irreducible modules for $U_q(\mathfrak{sl}_2)$.

**Lemma**

For all integers $d \geq 0$ and $\varepsilon \in \{1, -1\}$ there exists a $U$-module $V_{d,\varepsilon}$ with the following property: $V_{d,\varepsilon}$ has a basis $\{v_i\}_{i=0}^d$ such that

\[
kv_i = \varepsilon q^{d-2i} v_i \quad (0 \leq i \leq d), \\
fv_i = [i+1]_q v_{i+1} \quad (0 \leq i \leq d-1), \quad fv_d = 0, \\
ev_i = \varepsilon [d-i+1]_q v_{i-1} \quad (1 \leq i \leq d), \quad ev_0 = 0.
\]

The $U$-module $V_{d,\varepsilon}$ is irreducible provided that $q$ is not a root of unity.
Earlier we gave a Casimir element for $\Delta$. The algebra $U$ also has a Casimir element, which we now recall.

**Definition**

Define $\Lambda \in U$ as follows:

$$\Phi = ef(q - q^{-1})^2 + q^{-1}k + qk^{-1}.$$  

We call $\Lambda$ the (normalized) **Casimir element** of $U$.  

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The universal Askey-Wilson algebra
The following result is well known.

**Lemma**

The Casimir element $\Lambda$ is in the center $Z(U)$. Moreover on the $U$-module $V_{d,\varepsilon}$

$$\Lambda = \varepsilon(q^{d+1} + q^{-d-1})I.$$
The equitable presentation of $U_q(\mathfrak{sl}_2)$

When we defined $U$ we used the Chevalley presentation. There is another presentation for $U$ of interest, said to be equitable.

**Lemma (Tatsuro Ito, Chih-wen Weng, Ter 2000)**

*The algebra $U$ has a presentation by generators $x, y^{\pm 1}, z$ and relations*

\[
\begin{align*}
yy^{-1} &= y^{-1}y = 1, \\
\frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, \\
\frac{qyz - q^{-1}zy}{q - q^{-1}} &= 1, \\
\frac{qzx - q^{-1}xz}{q - q^{-1}} &= 1.
\end{align*}
\]

We call $x, y^{\pm 1}, z$ the **equitable generators** for $U$. 
In the equitable presentation the $U$-module $V_{d,\varepsilon}$ looks as follows.

$V_{d,\varepsilon}$ has three bases such that:

<table>
<thead>
<tr>
<th></th>
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<th>$y$</th>
<th>$z$</th>
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The Casimir element $\Lambda$ of $U_q(\mathfrak{sl}_2)$

In the equitable presentation of $U$ the Casimir element looks as follows.

**Lemma (Ter 2011)**

The Casimir element $\Lambda$ is equal to each of the following:

$$qx + q^{-1}y + qz - qxyz, \quad q^{-1}x + qy + q^{-1}z - q^{-1}zyx,$$
$$qy + q^{-1}z + qx - qyzx, \quad q^{-1}y + qz + q^{-1}x - q^{-1}xzy,$$
$$qz + q^{-1}x + qy - qzxy, \quad q^{-1}z + qx + q^{-1}y - q^{-1}yxz.$$
We are now ready to describe how $\Delta$ is related to $U_q(\mathfrak{sl}_2)$.

**Lemma (Ter 2011)**

Let $a, b, c$ denote nonzero scalars in $\mathbb{F}$. Then there exists an $\mathbb{F}$-algebra homomorphism $\Delta \to U_q(\mathfrak{sl}_2)$ that sends

\[
A \mapsto xa + ya^{-1} + \frac{xy - yx}{q - q^{-1}} bc^{-1},
\]

\[
B \mapsto yb + zb^{-1} + \frac{yz - zy}{q - q^{-1}} ca^{-1},
\]

\[
C \mapsto zc + xc^{-1} + \frac{zx - xz}{q - q^{-1}} ab^{-1}.
\]

The above homomorphism is not injective. To shrink the kernel we do the following.
From now on let $a, b, c$ denote mutually commuting indeterminates.

Let $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ denote the $\mathbb{F}$-algebra of Laurent polynomials in $a, b, c$ that have all coefficients in $\mathbb{F}$.

Consider the $\mathbb{F}$-algebra

$$U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}],$$

where $U = U_q(\mathfrak{sl}_2)$ and $\otimes = \otimes_{\mathbb{F}}$. 

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Theorem (Ter 2011)

There exists an injective $\mathbb{F}$-algebra homomorphism $\mathfrak{H} : \Delta \to U \otimes \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ that sends

- $A \mapsto x \otimes a + y \otimes a^{-1} + \frac{xy - yx}{q - q^{-1}} \otimes bc^{-1}$,
- $B \mapsto y \otimes b + z \otimes b^{-1} + \frac{yz - zy}{q - q^{-1}} \otimes ca^{-1}$,
- $C \mapsto z \otimes c + x \otimes c^{-1} + \frac{zx - xz}{q - q^{-1}} \otimes ab^{-1}$,

where $x, y, z$ denote the equitable generators for $U$. 
Theorem (Ter 2011)

The homomorphism \( \Delta \) sends

\[
\alpha \mapsto \Lambda \otimes (a + a^{-1}) + 1 \otimes (b + b^{-1})(c + c^{-1}), \\
\beta \mapsto \Lambda \otimes (b + b^{-1}) + 1 \otimes (c + c^{-1})(a + a^{-1}), \\
\gamma \mapsto \Lambda \otimes (c + c^{-1}) + 1 \otimes (a + a^{-1})(b + b^{-1}),
\]

where \( \Lambda \) denotes the Casimir element of \( U \).
Theorem (Ter 2011)

Under the homomorphism $\natural$ the image of the Casimir element $\Omega$ is

$$
1 \otimes (q + q^{-1})^2 - \Lambda^2 \otimes 1 - 1 \otimes (a + a^{-1})^2 - 1 \otimes (b + b^{-1})^2
- 1 \otimes (c + c^{-1})^2 - \Lambda \otimes (a + a^{-1})(b + b^{-1})(c + c^{-1})
$$

where $\Lambda$ denotes the Casimir element of $U$. 
Our next goal is to describe how $\Delta$ is related to the double affine Hecke algebra (DAHA) of type $(C^\vee_1, C_1)$.

This is the most general DAHA of rank 1.

We will work with the “universal” version of DAHA.

For notational convenience define a four element set

$$\mathbb{I} = \{0, 1, 2, 3\}.$$
The universal DAHA of type \((C_1^\vee, C_1)\)

**Definition**

Let \(\hat{\mathcal{H}}_q\) denote the \(\mathbb{F}\)-algebra defined by generators \(\{t_i^{\pm 1}\}_{i \in \mathbb{I}}\) and relations

\[
\begin{align*}
t_i t_i^{-1} &= t_i^{-1} t_i = 1 & & i \in \mathbb{I}, \\
&\quad \text{is central} & & i \in \mathbb{I}, \\
t_0 t_1 t_2 t_3 &= q^{-1}.
\end{align*}
\]

We call \(\hat{\mathcal{H}}_q\) the **universal DAHA of type** \((C_1^\vee, C_1)\).

For notational convenience define

\[
T_i = t_i + t_i^{-1} & & i \in \mathbb{I}.
\]
The elements $X, Y$ of $\hat{H}_q$

We will describe how $\Delta$ is related to $\hat{H}_q$.

To set the stage we first mention a few basic features of $\hat{H}_q$.

Define

$$X = t_3 t_0, \quad Y = t_0 t_1.$$  

Note that $X, Y$ are invertible.
A basis for $\hat{H}_q$

Theorem (Ter 2012)

The following is a basis for the $\mathbb{F}$-vector space $\hat{H}_q$:

$$Y^i X^j t_0^k T_1^r T_2^s T_3^t$$

$i, j, k \in \mathbb{Z} \quad r, s, t \in \mathbb{N}$.

This can be proven using the Bergman Diamond Lemma.
Corollary (Ter 2012)

The following are algebraically independent over $\mathbb{F}$:

$$t_0, \quad T_1, \quad T_2, \quad T_3.$$
We now describe the center $Z(\hat{H}_q)$.

**Theorem (Ter 2012)**

Assume that $q$ is not a root of unity. Then the algebra $Z(\hat{H}_q)$ is generated by $\{T_i\}_{i \in I}$.

Moreover $Z(\hat{H}_q)$ is isomorphic to a polynomial algebra in four variables.
Some automorphisms of $\hat{H}_q$

We mention some automorphisms of $\hat{H}_q$.

We start with an obvious one.

There exists an automorphism of $\hat{H}_q$ that sends

$$t_0 \mapsto t_1 \mapsto t_2 \mapsto t_3 \mapsto t_0.$$ 

We call this $\mathbb{Z}_4$-symmetry.

This symmetry sends

$$X \mapsto Y \mapsto q^{-1}X^{-1} \mapsto q^{-1}Y^{-1} \mapsto X.$$
We will be discussing the Artin braid group $B_3$.

**Definition**

The group $B_3$ is defined by generators $\rho, \sigma$ and relations $\rho^3 = \sigma^2$. For notational convenience define $\tau = \rho^3 = \sigma^2$.

There exists a group homomorphism $B_3 \to \text{PSL}_2(\mathbb{Z})$ that sends $\rho \mapsto p$ and $\sigma \mapsto s$. Via this homomorphism we pull back the $\text{PSL}_2(\mathbb{Z})$ action on $\Delta$, to get a $B_3$ action on $\Delta$ as a group of automorphisms.

Next we explain how $B_3$ acts on $\hat{H}_q$ as a group of automorphisms.
Lemma

The group $B_3$ acts on $\hat{H}_q$ as a group of automorphisms such that
$\tau(h) = t_0^{-1}ht_0$ for all $h \in \hat{H}_q$ and $\rho, \sigma$ do the following:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$t_0$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(h)$</td>
<td>$t_0$</td>
<td>$t_0^{-1}t_3t_0$</td>
<td>$t_1$</td>
<td>$t_2$</td>
</tr>
<tr>
<td>$\sigma(h)$</td>
<td>$t_0$</td>
<td>$t_0^{-1}t_3t_0$</td>
<td>$t_1t_2t_1^{-1}$</td>
<td>$t_1$</td>
</tr>
</tbody>
</table>
An action of $B_3$ on $\hat{H}_q$, cont.

Lemma

The $B_3$ action on $\hat{H}_q$ does the following to the central elements $\{T_i\}_{i \in \mathbb{I}}$. The generator $\tau$ fixes every central element. The generators $\rho, \sigma$ satisfy the table below.

<table>
<thead>
<tr>
<th></th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(h)$</td>
<td>$T_0$</td>
<td>$T_3$</td>
<td>$T_1$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>$\sigma(h)$</td>
<td>$T_0$</td>
<td>$T_3$</td>
<td>$T_2$</td>
<td>$T_1$</td>
</tr>
</tbody>
</table>

We are now ready to describe how $\Delta$ is related to $\hat{H}_q$. 
Theorem (Ter 2012)

There exists an injective \( \mathbb{F} \)-algebra homomorphism \( \psi : \Delta \to \hat{H}_q \) that sends

\[
A \mapsto t_1 t_0 + (t_1 t_0)^{-1}, \\
B \mapsto t_3 t_0 + (t_3 t_0)^{-1}, \\
C \mapsto t_2 t_0 + (t_2 t_0)^{-1}.
\]
Theorem (Ter 2012)

The homomorphism $\psi$ sends

- $\alpha \mapsto (q^{-1}t_0 + qt_0^{-1})(t_1 + t_1^{-1}) + (t_2 + t_2^{-1})(t_3 + t_3^{-1})$,
- $\beta \mapsto (q^{-1}t_0 + qt_0^{-1})(t_3 + t_3^{-1}) + (t_1 + t_1^{-1})(t_2 + t_2^{-1})$,
- $\gamma \mapsto (q^{-1}t_0 + qt_0^{-1})(t_2 + t_2^{-1}) + (t_3 + t_3^{-1})(t_1 + t_1^{-1})$. 
Theorem (Ter 2012)

Under the homomorphism $\psi$ the image of the Casimir element $\Omega$ is

$$(q + q^{-1})^2 - (q^{-1} t_0 + qt_0^{-1})^2 - (t_1 + t_1^{-1})^2 - (t_2 + t_2^{-1})^2$$
$$- (t_3 + t_3^{-1})^2 - (q^{-1} t_0 + qt_0^{-1})(t_1 + t_1^{-1})(t_2 + t_2^{-1})(t_3 + t_3^{-1}).$$
Theorem (Ter 2012)

For all $g \in B_3$ the following diagram commutes:

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\psi} & \hat{H}_q \\
g \downarrow & & \downarrow g \\
\Delta & \xrightarrow{\psi} & \hat{H}_q \\
\end{array}
\]
Now consider the image of $\Delta$ under $\psi$.

As we will see, this image is related to the "spherical subalgebra"

$$\{ h \in \hat{H}_q \mid t_0 h = h t_0 \}. $$
Consider the image of $\Delta$ under $\psi$. The spherical subalgebra
\[ \{ h \in \hat{H}_q \mid t_0 h = h t_0 \} \] is generated by this image together with $t_0^{\pm 1}, T_1, T_2, T_3$. 
For notational convenience, from now on identify $\Delta$ with its image under the injection $\psi : \Delta \to \hat{H}_q$.

From this point of view

\[
A = t_1 t_0 + (t_1 t_0)^{-1} = t_0 t_1 + (t_0 t_1)^{-1} = Y + Y^{-1},
\]
\[
B = t_3 t_0 + (t_3 t_0)^{-1} = t_0 t_3 + (t_0 t_3)^{-1} = X + X^{-1},
\]
\[
C = t_2 t_0 + (t_2 t_0)^{-1} = t_0 t_2 + (t_0 t_2)^{-1},
\]

\[
\alpha = (q^{-1} t_0 + q t_0^{-1}) T_1 + T_2 T_3,
\]
\[
\beta = (q^{-1} t_0 + q t_0^{-1}) T_3 + T_1 T_2,
\]
\[
\gamma = (q^{-1} t_0 + q t_0^{-1}) T_2 + T_3 T_1,
\]

\[
\Omega = (q + q^{-1})^2 - (q^{-1} t_0 + q t_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2
\]
\[
- (q^{-1} t_0 + q t_0^{-1}) T_1 T_2 T_3.
\]
We now give a presentation of the spherical subalgebra \( \{ h \in \hat{H}_q \mid t_0 h = h t_0 \} \) by generators and relations.

This will be our last result of the talk.
The spherical subalgebra \( \{ h \in \hat{H}_q \mid t_0 h = h t_0 \} \) is presented by generators and relations in the following way. The generators are \( A, B, C, t_0^{\pm 1}, \{ T_i \}_{i=1}^{3} \). The relations assert that each of \( t_0^{\pm 1}, \{ T_i \}_{i=1}^{3} \) is central and \( t_0 t_0^{-1} = 1, t_0^{-1} t_0 = 1 \),

\[
\begin{align*}
A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{\alpha}{q + q^{-1}}, \\
B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{\beta}{q + q^{-1}}, \\
C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{\gamma}{q + q^{-1}}, \\
q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma &= (q + q^{-1})^2 - (q^{-1}t_0 + qt_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2 - (q^{-1}t_0 + qt_0^{-1}) T_1 T_2 T_3,
\end{align*}
\]
A presentation for the spherical subalgebra, cont.

**Theorem**

where

\[ \alpha = (q^{-1} t_0 + qt_0^{-1}) T_1 + T_2 T_3, \]
\[ \beta = (q^{-1} t_0 + qt_0^{-1}) T_3 + T_1 T_2, \]
\[ \gamma = (q^{-1} t_0 + qt_0^{-1}) T_2 + T_3 T_1. \]
In this talk we introduced the universal Askey-Wilson algebra $\Delta$. We showed how each Leonard pair and Leonard triple of QRacah type yields a $\Delta$-module.

We discussed how $\Delta$ is related to $Q$-polynomial distance-regular graphs of QRacah type.

We gave several bases for $\Delta$, we described its center, and we showed how $\text{PSL}_2(\mathbb{Z})$ acts on $\Delta$ as a group of automorphisms.

We described how $\Delta$ is related to $U_q(\mathfrak{sl}_2)$.

Finally we described how $\Delta$ is related to the universal DAHA of type $(C_1^\vee, C_1)$.

Thank you for your attention!

THE END


P. Terwilliger. The universal Askey-Wilson algebra and DAHA of type $(C_1^\vee, C_1)$. Submitted for publication.